Logic, Computation, and the Expressive Power of the Modal μ-Calculus

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Introduction

Motivation

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Transition systems model the behaviour of programs...
...hence it would be useful to state and verify properties of transition systems

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Motivations

Ok, but what is a transition system?

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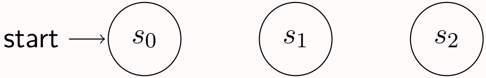
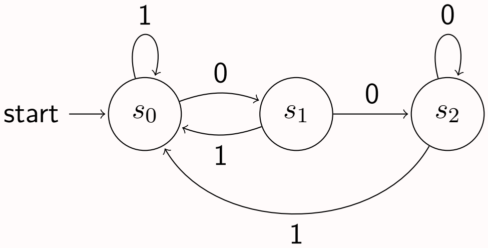
Transition systems

A transition system is a set of states



Transition systems

A transition system is a set of states, with rules about how to go from one state to another.



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Transition systems

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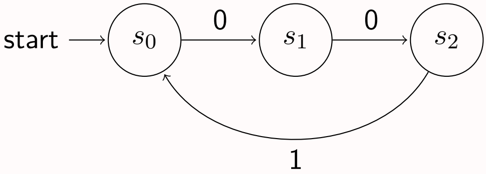
If $(s, a, t) \in \rightarrow$, we write $s \xrightarrow{a} t$. So in the following transition system, $S = \{s_0, s_1, s_2\}$, $A = \{0, 1\}$ and $s_0 \xrightarrow{0} s_1$, $s_1 \xrightarrow{0} s_2$ and $s_2 \xrightarrow{1} s_0$.



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Transition systems

We can enrich the states by assigning propositions to them, via a function $D : AP \to 2^S$, where AP is a set of *atomic propositions*. D maps a proposition to the set of states at which that proposition is true.

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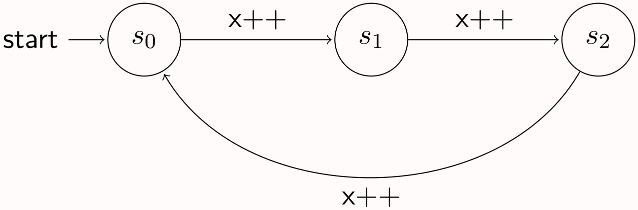
Example

Mod 3 counter:



$$A = \{x{+}{+}\}, \quad AP = \{x = 0, x = 1, x = 2\}$$

$$D(x = 0) = \{s_0\},\ D(x = 1) = \{s_1\},\ D(x = 2) = \{s_2\}$$

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Using logic to talk about transition systems

Suppose we asked someone to make us a mod 3 counter, and we were given a program whose transition system is the one on the previous slide. How can we test that the program we received is correct?

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Specifically, we would give a *specification* of what we want, and we want to check the actual program fits this specification. A suitable logic gives us a way to specify the properties we want our program to have.

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Syntax: what we can write down
Semantics: what it means

There are many logics, and we are interested in the ones which allow us to talk about transition systems.

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$$\varphi, \psi ::= P \mid \varphi \wedge \psi \mid \neg\varphi$$

Where $P \in AP$. We can use *de Morgan duality* to define $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$, and define $\varphi \implies \psi$ as $\psi \vee \neg\varphi$.

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So if $AP = \{p, q\}$, then $p \wedge q$ and $\neg q$ are allowable, but $p \vee q \wedge$ or $p \neg \wedge p$ are not.

$*$

Using logic to talk about transition systems

As for semantics, we can define what a logical formula means in terms of transition systems by declaring when a state *satisfies* a formula. If s is a state and φ a formula, we write $s \models \varphi$ to say that the state s satisfies φ, i.e. φ is true at s. Hence we can define the semantics of a formula by assigning to a formula a set of states at which it holds.

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We will inductively define $[\![\varphi]\!]$ as the set of states at which φ holds, thus giving meaning to our formulas in terms of states of a transition system as follows:

$$[\![P]\!] = D(P)$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![\neg\varphi]\!] = S \setminus [\![\varphi]\!]$$

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Then we may say $s \models \varphi$ if $s \in [\![\varphi]\!]$.

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Using logic to talk about transition systems

Given these semantics, we can use the logic we've outlined above to make a specification for our mod 3 counter. For example, in our specification, we might require that x, the counter, is always 0, 1, or 2. Given our atomic propositions, we can write a formula expressing this: $\varphi_1 := (x = 0) \vee (x = 1) \vee (x = 2)$. Then we can check that $[\![\varphi_1]\!] = \{s_0, s_1, s_2\}$.

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Calculating:

$$\begin{aligned} [\![\varphi_1]\!] &= [\![(x = 0) \vee (x = 1) \vee (x = 2)]\!] \\ &= [\![x = 0]\!] \cup [\![x = 1]\!] \cup [\![x = 2]\!] \\ &= D(x = 0) \cup D(x = 1) \cup D(x = 2) \\ &= \{s_0\} \cup \{s_1\} \cup \{s_2\} = \{s_0, s_1, s_2\} \end{aligned}$$

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We cannot specify other properties we might require. For example, we might want to say that if the action $x{+}{+}$ occurs in a state where $x = 2$ is true, then the action takes us to a state where $x = 0$ is true. We cannot write this with our current logic.

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This is where a modal logic proves useful. Briefly, modal logics include operators called modalities which allow us to qualify statements. Often there are modalities $\diamond$ and $\square$, which express dual notions analogous to "possibly" and "necessarily".

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Using logic to talk about transition systems

We can use these in a number of ways to talk about transition systems. For example, if we restrict ourselves to talking about a particular path in a transition system (say, a path P in the directed graph representing a transition system), then we could interpret $\diamond\varphi$ as "φ is true at some point along P" and $\Box\varphi$ as "φ is true at every point along P".

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Or we can consider the entire transition system at once, and label modalities with actions $a \in A$, so that given a state s, $\langle a\rangle\varphi$ means "there exists an a transition out of s to a state where φ is true" and $[a]\varphi$ means "every a transition out of s goes to a state where φ is true".

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We will look at such logics more precisely later in this talk.

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Given a partially ordered set $(L, \leq)$, and a subset $A \subseteq L$, an element $u \in L$ is an upper bound for A if $a \leq u$ for all $a \in A$. A *least* upper bound for A is an upper bound l such that $l \leq u$ for all upper bounds u of A. In lattice-theoretic terms, we call the least upper bound a join.

Lattices and the Knaster-Tarski theorem

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If every two-element subset $\{a, b\} \subseteq L$ has a meet and join, denoted $a \wedge b$ and $a \vee b$ respectively, L is called a *lattice*.

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If every subset $A \subseteq L$ has a meet and a join, L is a *complete* lattice. This implies there is an element at the "top" of the lattice, $\top = \bigwedge L$ and one at the "bottom" of the lattice $\bot = \bigvee L$.

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Given a set X, the powerset $(2^X, \subseteq)$ is a classic example of a complete lattice, with $\top = X$ and $\bot = \varnothing$. The meet is intersection ($\wedge = \cap$) and join is union ($\vee = \cup$).

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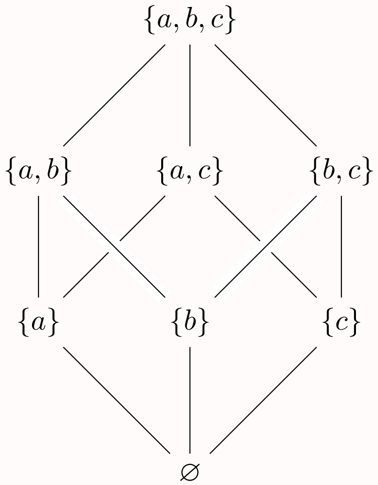
Lattices and the Knaster-Tarski theorem



Lattices and the Knaster-Tarski theorem

A function $f : L \to L$ is *order-preserving* (or *monotone*) if $x \leq y \implies f(x) \leq f(y)$. An element $x \in L$ is a *fixed point* of f if $f(x) = x$. With these definitions in place, we are ready to state a very cool theorem with an appropriately cool name:

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Theorem. (*Knaster-Tarski*) If L is a complete lattice and $f : L \to L$ is an order-preserving function, then the set of fixed points of f forms a complete lattice (in particular, there exists a least fixed point μf and a greatest fixed point νf).

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Ok now that you've learned 2 semesters of computer science theory, let's get cracking.

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The modal μ-calculus $L\mu$

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In 1983, Dexter Kozen introduced the modal μ-calculus $L\mu$, which enhances a simple syntax with powerful fixed-point operators and subsumes the logics above.

Today we will show that $L\mu$ subsumes PDL in particular. The goal is to show that $L\mu$ is **strictly** more expressive than PDL.

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Syntax of $L\mu$

$$\varphi, \psi ::= P$$

Atomic propositions $P \in AP$. Includes $\top$

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$$\varphi, \psi ::= P \mid X \mid \varphi \wedge \psi \mid \neg\varphi \mid [a]\varphi \mid \nu X.\varphi(X)$$

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We have an additional syntactic constraint on $\varphi(X)$ in $\nu X.\varphi(X)$: X must be free in φ and occur *positively* - in the scope of an even number of negations.

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We have an additional syntactic constraint on $\varphi(X)$ in $\nu X.\varphi(X)$: X must be free in φ and occur *positively* - in the scope of an even number of negations.
The other usual operators can be obtained by de Morgan duality:

$$\bot \equiv \neg\top$$

$$\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$$

$$\langle a\rangle\varphi \equiv \neg[a]\neg\varphi$$

$$\mu X.\varphi(X) \equiv \neg\nu X.\neg\varphi(\neg X)$$

*

Semantics of $L\mu$

We can define the semantics of $L\mu$ in terms of states of a transition system TS over a set of states S, where we have a function $D : AP \to 2^S$ mapping atomic propositions to the states at which they hold ($D(\top) = S$). We define $[\![\varphi]\!]$, the set of all states satisfying φ, inductively as follows:

$$[\![P]\!] = D(P)$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$

$$[\![\neg\varphi]\!] = S \setminus [\![\varphi]\!]$$

$$[\![[a]\varphi]\!] = \{s \in S \,|\, \forall t \,.\, s \xrightarrow{a} t \implies t \in [\![\varphi]\!]\}$$

$$[\![\langle a\rangle\varphi]\!] = \{s \in S \,|\, \exists t \,.\, s \xrightarrow{a} t \wedge t \in [\![\varphi]\!]\}$$

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Semantics of $L\mu$

If a formula contains a variable X, we interpret $[\![\varphi(X)]\!]$ as a function $T \mapsto [\![\varphi[T/X]]\!]$ mapping sets of states $T \subseteq S$ to an interpretation of φ where all instances of X have been replaced by the states in T. We interpret this mixing of formulas and states like this (for example):

$$s \in [\![\psi \wedge T]\!] \text{ if } s \in [\![\psi]\!] \text{ and } s \in T$$

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For notational simplicity we will consider formulas of a single variable, and write $[\![\varphi(\psi)]\!]$ to express $[\![\varphi(X)]\!]([\![\psi]\!])$.

$*$

Semantics of $L\mu$

Formulas $\varphi(X)$ that obey the positivity restriction define monotonic functions $[\![\varphi(X)]\!] : 2^S \to 2^S$ on the powerset lattice, which is complete. Hence we can define $[\![\mu X.\varphi(X)]\!]$ and $[\![\nu X.\varphi(X)]\!]$ to be the least and greatest fixed points of $[\![\varphi(X)]\!]$.

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Recursion semantics

Lattice theory tells us that monotone functions f mapping a complete lattice to itself have fixed points, which is how we defined the semantics of the formulas $\nu X.\varphi(X)$ and $\mu X.\varphi(X)$.

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Furthermore, we may obtain these fixed points by successive iterations of f. For instance, $\mu f = \bigvee_n f^n(\bot)$

Hence the phrase "started from the bottom now we're here"

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Recursion semantics

$$\mu f = \bigvee_n f^n(\bot) \quad \rightsquigarrow \quad [\![\mu X.\varphi(X)]\!] = \bigcup_n [\![\varphi^n(\bot)]\!]$$

This iteration will take at most $|S| + 1$ powers of φ to reach the fixed point.

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$$[\![\bot]\!] \subseteq [\![\varphi(\bot)]\!] \subseteq [\![\varphi(\varphi(\bot))]\!] \subseteq \ldots \subseteq [\![\varphi^n(\bot)]\!] \subseteq \ldots$$

If the fixed point is at some power n, then there is a finite increasing chain of sets of states which satisfy $\mu X.\varphi(X)$.

"μ is finite looping"

$$*$$

Example

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$$\begin{aligned} [\![[a]\bot]\!] &= \{ s \in S \mid \forall t \,.\, s \xrightarrow{a} t \implies t \in [\![\bot]\!] \} \\ &= \text{set of states with no outgoing } a \text{ transitions} \end{aligned}$$

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$$= \text{set of states with no outgoing } a \text{ transitions}$$

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$$\llbracket [a][a]\bot \rrbracket = \{s \in S \mid \forall t \,.\, s \xrightarrow{a} t \implies t \in \llbracket [a]\bot \rrbracket\}$$
$$= \text{set of states whose } a \text{ transitions go}$$
$$\text{to states with no } a \text{ transitions}$$

(all a paths are length 1)

Example

What does this express?

$$\mu X.[a]X$$

And so on. If a state s is in $[\![\mu X.[a]X]\!]$, then all a paths starting at s are finite.
We can say $TS \models \varphi$ if every initial state s_0 is in $[\![\varphi]\!]$.
Hence $TS \models \mu X.[a]X$ if TS contains no infinite initial a paths.

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Propositional Dynamic Logic

Introduction to PDL

Propositional Dynamic Logic is another modal logic. Labels on modalities like $\langle\alpha\rangle$ and $[\alpha]$ represent (non-deterministic) programs, and we read formulas with these modalities as:

$\langle\alpha\rangle\varphi \quad \mapsto \quad$ "**Some** terminating execution of α ends in a state satisfying φ"

$[\alpha]\varphi \quad \mapsto \quad$ "**Every** execution of α leads to a state satisfying φ"

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Introduction to PDL

If we give ourselves a set of basic atomic programs $a, b, \ldots$ which go from state to state, we can write more complex programs with the familiar operations in regular expressions.

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$\alpha;\ \beta$: Sequentially execute α, then β

α^* : Execute α some finite number of times (perhaps 0)

$*$

Syntax of PDL

Formulas in PDL follow the usual syntax

$$\varphi, \psi ::= P \mid \varphi \wedge \psi \mid \neg\varphi \mid [\alpha]\varphi$$

We can obtain $\langle\alpha\rangle\varphi$ and $\varphi \vee \psi$ by taking the de Morgan dual as usual.

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Formulas express properties of states in transition systems, so we may make judgements such as $s \models \varphi$ for some state s, and extend the satisfaction relation to transition systems, such that $TS \models \varphi$ if every initial state $s_0 \models \varphi$.

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Small Model Property

PDL (like the other logics mentioned earlier) has the **small model property**, which means that if φ is satisfiable, i.e. if there is a transition system TS such that $TS \models \varphi$, then there is a finite transition system TS_{FIN} such that $TS_{FIN} \models \varphi$.

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Small Model Property

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In this way, we get a usable method to transform transition systems satisfying φ into other, finite transition systems.

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Formally, because this is both cool and crucial to our main result: suppose $TS \models \varphi$.

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Let Γ be the set of all sub-formulas of φ and their negations; Γ is finite. Define an equivalence relation $\sim$ on the states S in TS such that $s \sim t$ if for all $\psi \in \Gamma$, $s \models \psi \iff t \models \psi$.

Small Model Property

Formally, because this is both cool and crucial to our main result: suppose $TS \models \varphi$.

Let Γ be the set of all sub-formulas of φ and their negations; Γ is finite. Define an equivalence relation $\sim$ on the states S in TS such that $s \sim t$ if for all $\psi \in \Gamma$, $s \models \psi \iff t \models \psi$.

There are at most $2^{|\Gamma|}$ equivalence classes in $S/\sim$ (2 possible truth values for each sub-formula); if we let $[s], [t] \in S/\sim$ represent states in a new TS_{FIN}, with $[s] \xrightarrow{a} [t]$ if for some $s' \in [s]$ and $t' \in [t]$, $s' \xrightarrow{a} t'$, then one can show TS_{FIN} also satisfies φ.

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Expressing PDL in $L\mu$

Expressing the modalities

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Verifying these formulas are equivalent is an exercise in semantics; let's look at the most interesting case:

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Using our iteration again, $[\![\varphi \vee \langle \alpha \rangle \bot]\!]$ is the set of all states satisfying φ (no states satisfy $\langle \alpha \rangle \bot$). Then $[\![\varphi \vee \langle \alpha \rangle (\varphi \vee \langle \alpha \rangle \bot)]\!]$ is the set of all states which either satisfy φ, or in which there is a α transition to a state satisfying φ.

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Iterating this, $s \models \mu X.\varphi \vee \langle \alpha \rangle X$ if and only if there is an α path from s reaching a state satisfying φ. This is precisely the condition defining $\langle \alpha^* \rangle \varphi$.

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Showing $L\mu$ is strictly more expressive

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Suppose φ is a PDL formula which is equivalent to $\mu X.[a]X$. Then if $TS \models \mu X.[a]X$, $TS \models \varphi$ as well.

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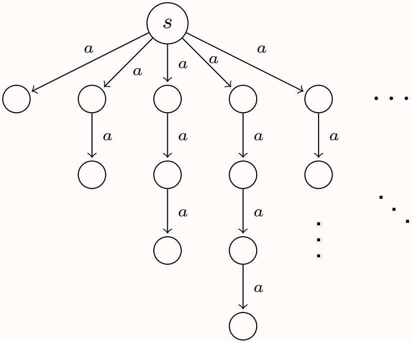
Showing $L\mu$ is strictly more expressive

Consider the following transition system TS with initial state s:



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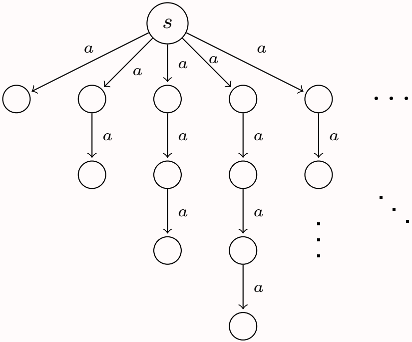

Every path from s is finite length, hence $TS \models \mu X.[a]X$.

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Showing $L\mu$ is strictly more expressive

If φ (the PDL formula) is equivalent to $\mu X.[a]X$, then $TS \models \varphi$ as well.

By the proof of the small model property, we can then collapse TS to a finite TS_{FIN} which also satisfies φ. Since $\varphi \equiv \mu X.[a]X$, it follows that $TS_{FIN} \models \mu X.[a]X$.

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But TS_{FIN} must contain a loop as a result of the filtration process, so there is an infinite a path. This gives a contradiction.

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Showing $L\mu$ is strictly more expressive

So there is no PDL formula equivalent to $\mu X.[a]X$, and $L\mu$ is strictly more expressive than PDL.

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Thank you for your time! Questions?

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